Solving Diophantine Equations

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Topics Covered This Semester

- Elementary Methods
  - The Decomposition Method
  - Solving Using Inequalities
  - The Modular Arithmetic Method
  - Inductive Methods
  - Fermat's Method of Infinite Descent
- Some Classical Diophantine Equations
  - Linear Diophantine Equations
  - Pythagorean Triples and Related Problems
- Pell's Type Equations
  - History and Motivation
  - Deriving the Formula
  - Solving Using Elementary Methods

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Outline

Part I

What is a Diophantine Equation?

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- $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{n}$, for some integer n
- $u^2 - Dv^2 = 1$, for some integer D

Part II

Methods of Solving

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We get four systems:

$$\begin{cases} x + 2y = 1 \\ x + 4y + 3 = 2 \end{cases} ; \begin{cases} x + 2y = 2 \\ x + 4y + 3 = 1 \end{cases}$$

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Solutions: $\{(3,-1), (6,-2), (3,-2), (0,-1)\}$.

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- Since no two residues from these distinct lists sum to $7 \pmod{13}$, $x^3 + y^4 \not\equiv 7 \pmod{13}$.

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No solutions.

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Theorem (Pell's Type Equation Solutions)

The equation $u^2 - Dv^2 = 1$, where D is not a perfect square, has infinitely many solutions given by:

- $u_n = \frac{1}{2}\left[(u_0 + v_0\sqrt{D})^n + (u_0 - v_0\sqrt{D})^n\right]$
- $v_n = \frac{1}{2\sqrt{D}}\left[(u_0 + v_0\sqrt{D})^n - (u_0 - v_0\sqrt{D})^n\right]$,

where (u_0, v_0) is the smallest nontrivial solution ("fundamental" solution).

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- Sides: $n-1, n, n+1$
- By Heron's formula, Area $= \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$, and a, b, c are the sidelengths.

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- $\implies A = \frac{n\sqrt{3(n^2-4)}}{4}$
- If n were odd, A would not be an integer, so $n = 2x$.
- We also need $3(n^2-4)$ to be a perfect square, so $n^2 - 4 = 3z^2$.
  - $\implies z$ is even, so $z = 2y$.
- Then, $4x^2 - 4 = 12y^2$, or $x^2 - 3y^2 = 1$.

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Apply the Pell's equation formula to $x^2 - 3y^2 = 1$ with fundamental solution $(2, 1)$:

- $x_n = \frac{1}{2}\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n\right]$
- $y_n = \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right]$

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Apply the Pell's equation formula to $x^2 - 3y^2 = 1$ with fundamental solution $(2, 1)$:

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- $y_n = \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right]$

The sides are $2x_n - 1, 2x_n, 2x_n + 1$ and the areas are $A = 3x_n y_n$.

References

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An Introduction to Diophantine Equations.
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