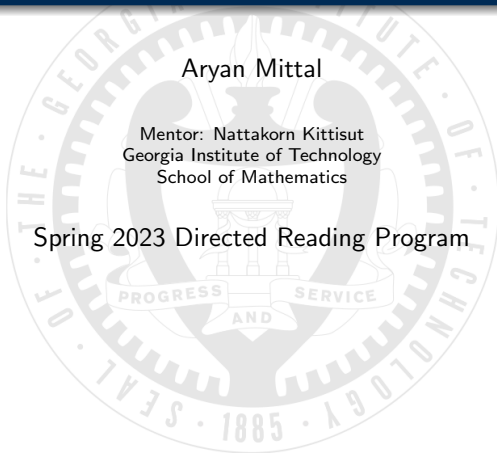


# Solving Diophantine Equations

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Spring 2023 Directed Reading Program



# Topics Covered This Semester

- Elementary Methods
  - The Decomposition Method
  - Solving Using Inequalities
  - The Modular Arithmetic Method
  - Inductive Methods
  - Fermat's Method of Infinite Descent
- Some Classical Diophantine Equations
  - Linear Diophantine Equations
  - Pythagorean Triples and Related Problems
- Pell's Type Equations
  - History and Motivation
  - Deriving the Formula
  - Solving Using Elementary Methods

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# Outline

- I What is a Diophantine Equation?
- II Methods of Solving

## Part I

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- $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{n}$ , for some integer  $n$
- $u^2 - Dv^2 = 1$ , for some integer  $D$

## Part II

# Methods of Solving

- 1 The Decomposition Method
- 2 The Modular Arithmetic Method
- 3 Pell's Type Equations

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**Solutions:**  $\{(3, -1), (6, -2), (3, -2), (0, -1)\}$ .

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**No solutions.**

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## Theorem (Pell's Type Equation Solutions)

*The equation  $u^2 - Dv^2 = 1$ , where  $D$  is not a perfect square, has infinitely many solutions given by:*

- $u_n = \frac{1}{2} \left[ (u_0 + v_0\sqrt{D})^n + (u_0 - v_0\sqrt{D})^n \right]$
- $v_n = \frac{1}{2\sqrt{D}} \left[ (u_0 + v_0\sqrt{D})^n - (u_0 - v_0\sqrt{D})^n \right],$

*where  $(u_0, v_0)$  is the smallest nontrivial solution ("fundamental" solution).*

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- $\implies A = \frac{n\sqrt{3(n^2-4)}}{4}$
- If  $n$  were odd,  $A$  would not be an integer, so  $n = 2x$ .
- We also need  $3(n^2 - 4)$  to be a perfect square, so  $n^2 - 4 = 3z^2$ .

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- If  $n$  were odd,  $A$  would not be an integer, so  $n = 2x$ .
- We also need  $3(n^2 - 4)$  to be a perfect square, so  $n^2 - 4 = 3z^2$ .
  - $\implies z$  is even, so  $z = 2y$ .
- Then,  $4x^2 - 4 = 12y^2$ , or  $x^2 - 3y^2 = 1$ .

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- $x_n = \frac{1}{2} [(2 + \sqrt{3})^n + (2 - \sqrt{3})^n]$
- $y_n = \frac{1}{2\sqrt{3}} [(2 + \sqrt{3})^n - (2 - \sqrt{3})^n]$

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**The sides are  $2x_n - 1, 2x_n, 2x_n + 1$  and the areas are  $A = 3x_n y_n$ .**



# References



Titu Andreescu and Dorin Andrica.  
*An Introduction to Diophantine Equations.*  
GIL Publishing House, 2002.