Gödel's Incompleteness Theorems

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CS 4510 Final Project

Outline

Part I

History

Foundational Crisis of Mathematics

- Establishing the Foundations of Math
  - 300 BC: Euclid's *Elements*
  - 1874: Cantor's set theory
  - 1879: Frege's *Begriffsschrift*
- Russell's Paradox: A Contradiction
  - $S = \{x \mid x \notin x\}$ ("The set of all sets that don't contain themselves")
  - Is $S \in S$?
  - Contradiction both ways!
- New Field: Formal Logic
  - 1910, 1912, 1914: Russell and Whitehead's *Principia Mathematica*

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Hilbert's Program

1. Completeness
2. Consistency
3. Decidability

Hilbert's Program

1. Completeness
2. Consistency
3. ~~Decidability~~
   - Turing's Halting Problem

Hilbert's Program

1. ~~Completeness~~
   - Gödel's First Incompleteness Theorem
2. Consistency?
   - Gödel's Second Incompleteness Theorem
3. ~~Decidability~~

Part II

Background and Definitions

Formal Systems

Definition (Formal System)

A *formal system* is a system of axioms equipped with rules of inference, which allow one to generate new theorems (e.g. ZFC).

Fundamental Theorem of Arithmetic

Theorem (Fundamental Theorem of Arithmetic)

Every natural number has a unique prime factorization.

Gödel Numberings

A **Gödel numbering** associates logical statements to unique natural numbers.
To define this, first we map each mathematical symbol in our formal system to a number.

x	N(x)
0	1
=	2
¬	3
(	4
)	5
⋮	⋮

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Gödel Numberings

Definition (Gödel Numbering Γ of a statement $f = f_1 f_2 \ldots f_n$)

$$\Gamma(f) = \prod_{i=1}^{n} p_i^{N(f_i)},$$

where p_i is the i^{th} prime and $N(f_i)$ is the number associated to symbol f_i by the chosen mapping.

Example

Consider the statement $f =$ "$0 = 0$".
We first map each symbol to a number to get $< 1, 2, 1 >$.
From here, we calculate the Gödel number as $\Gamma(f) = 2^1 3^2 5^1 = 90$.

Gödel Numberings

Why convert statements to numbers?

- To prove properties of formal systems via the known properties of number theory
- Associate each logical operation on statements f_1 and f_2 with an arithmetic operation on $\Gamma(f_1)$ and $\Gamma(f_2)$
  - Gödel proved the correctness of 46 of these numerical operations
  - Essentially created a computer to do math using number theory
- For example, **Sub** corresponds to dividing out and multiplying in the appropriate prime powers

Common Notation

- x **Sub** (u, y): within the statement associated with a number x, whenever you see a u substitute a y
- $p \vdash A$: p proves some statement A in the language of p

Part III

The 1st Incompleteness Theorem

The 1st Incompleteness Theorem

The Statement

Theorem

There cannot exist a formal system capable of sufficient arithmetic (i.e. not trivial) that is both consistent and complete.

The 1st Incompleteness Theorem

Proof Idea

- Contradictions seem to arise from self-referential statements
  - E.g. “This statement is false”, set of all sets that don’t contain themselves, etc.
- Encode mathematical statements, then use the encoding recursively to produce a self-referential statement
- Can we produce a statement talking about its own provability?

The Proof

- Consider the formula $f(x) = \neg\exists\, p\ [\ p \vdash (x\ \textbf{Sub}\ (0, x))\ ]$
  - Here, f takes as input a Gödel number x of a statement.
  - "There does not exist a proof p such that p proves x substituted for each instance of 0 with x."
- This formula has Gödel number $\Gamma(f)$.
- Pass $\Gamma(f)$ in as input to f:

$$f(\Gamma(f)) = \neg\exists\, p\ [\ p \vdash (\Gamma(f)\ \textbf{Sub}\ (0, \Gamma(f)))\ ]$$

- Simplify the inside $\Gamma(f)\ \textbf{Sub}\ (0, \Gamma(f))$:
  - $\Gamma(f)\ \textbf{Sub}\ (0, \Gamma(f))$
  - $= \neg\exists\, p\ [\ p \vdash (\Gamma(f)\ \textbf{Sub}\ (0, \Gamma(f)))\ ]$
  - $= f(\Gamma(f))$
- Hence, $f(\Gamma(f)) = \neg\exists\, p\ [\ p\ \vdash f(\Gamma(f))\ ]$.

The Proof

- If we name the statement $g = f(\Gamma(f))$ for ease, we have $g = \neg\exists\, p\, [\, p \vdash g\,]$.
  - g says "There does not exist a proof p of g" $\iff$ "This statement is unprovable"
- If g is false, then there exists a proof of it, but that would make it true, a contradiction.
- g must be true and unprovable.

$\square$

Notes on the Proof

- Generally interpret this theorem to mean "Every consistent system has unprovable true statements."
- Can't just add unprovable statements as axioms

Part IV

The 2nd Incompleteness Theorem

The 2nd Incompleteness Theorem

The Statement

Theorem

Any formal system capable of sufficient arithmetic cannot prove its own consistency.

The 2nd Incompleteness Theorem

The Proof

- Assume (for contradiction) that there exists inside formal system F a proof C of F's own consistency.
- Recall our Gödel sentence g ("This statement is unprovable") from the proof of the first theorem.
- The first theorem showed that if a system is consistent, then g is unprovable within it.
- By definition, $C \implies [F \text{ is consistent}]$.
- But, by the first incompleteness theorem, $[F \text{ is consistent}] \implies g$.
- This is a proof of g, but g was already shown to be unprovable.
- Hence, the mere existence of C, a proof of F's own consistency, leads to a contradiction, so such a proof cannot exist.

□

Notes on the Proof

Why is g being true an issue/contradiction here, but not in the first theorem?

- A contradiction arises when g is *proven*.
- The first theorem said "if F is consistent, then g is true."
- It doesn't become a proof of g until you prove that F is consistent.

Part V

Implications and Related Results

Implications

- Cannot create a system containing all truths and their proofs
- No system can verify its own reliability

Related Results in Consistency

- Gentzen's Consistency Proof
  - Proved the Peano Axioms are consistent
  - Proof relies on another system being consistent
- Why can't we prove ZFC's consistency?
  - Almost all math expressible in ZFC
  - Need to step out of ZFC to prove something about it
  - Not enough math in systems stronger than ZFC

Related Results in Unprovability

- Statements can be true, false, or unprovable
- Examples of proven unprovable claims
  - Paris-Harrington Theorem (first)
  - Kruskal's Theorem
  - Goodstein's Theorem
- Continuum Hypothesis
  - We know $|\mathbb{N}| < |\mathbb{R}|$
  - Is there a set $\mathcal{S}$ such that $|\mathbb{N}| < |\mathcal{S}| < |\mathbb{R}|$?
  - Unprovable within ZFC and truth value still unknown

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